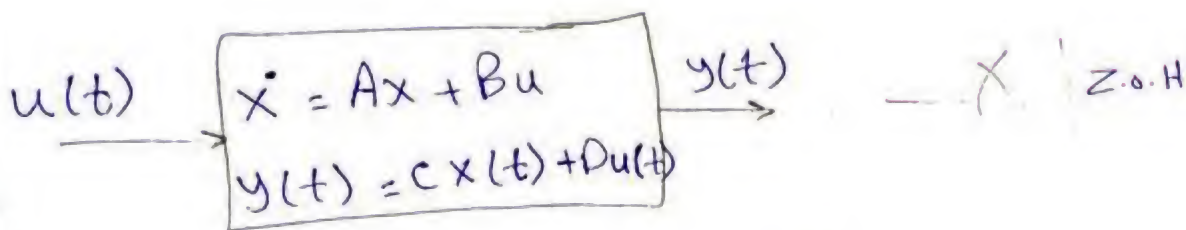


Digital Control

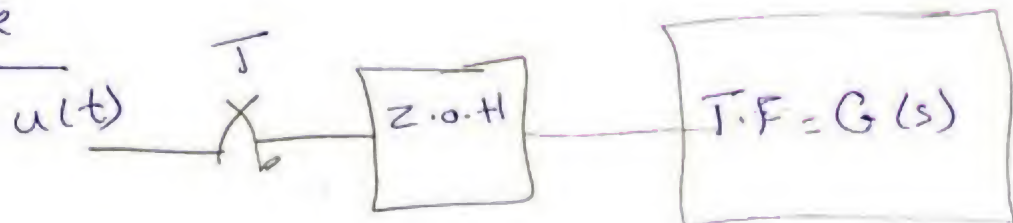
* The corresponding state-space representation of a continuous system in the discrete-time domain:-

In continuous time system



$$T.F = \frac{Y(s)}{U(s)} = sC(sI - A)^{-1}B + D$$

In discrete



$$x(k+1) = A_d x(kT) + B_d u(t)$$

$$y(kT) = C_d x(kT) + D_d u(t)$$

$$\text{Pulse T.F} = \sum_{k=0}^{\infty} \left[\frac{1 - e^{-Ts}}{s} \cdot G(s) \right] \quad \text{T.F.}$$

* Prove the following

$$A_d = \phi(\tau)$$

$$B_d = \int_0^T \phi(\eta) B d\eta$$

$$C_d = C$$

$$D_d = D$$

Given

$$\dot{x}(t) = A x(t) + B u(t)$$

$$y(t) = C x(t) + D u(t)$$

sampler + Z.O.H
↳ discretization

$$x(k+1)T = A_d x(kT) + B_d u(kT)$$

$$y(kT) = C_d x(kT) + D_d u(kT)$$

$$y(t) = C x(t) + D u(t) \xrightarrow{t=kT} y(kT) = C_d x(kT) + D_d u(kT)$$

$$C = C_d$$

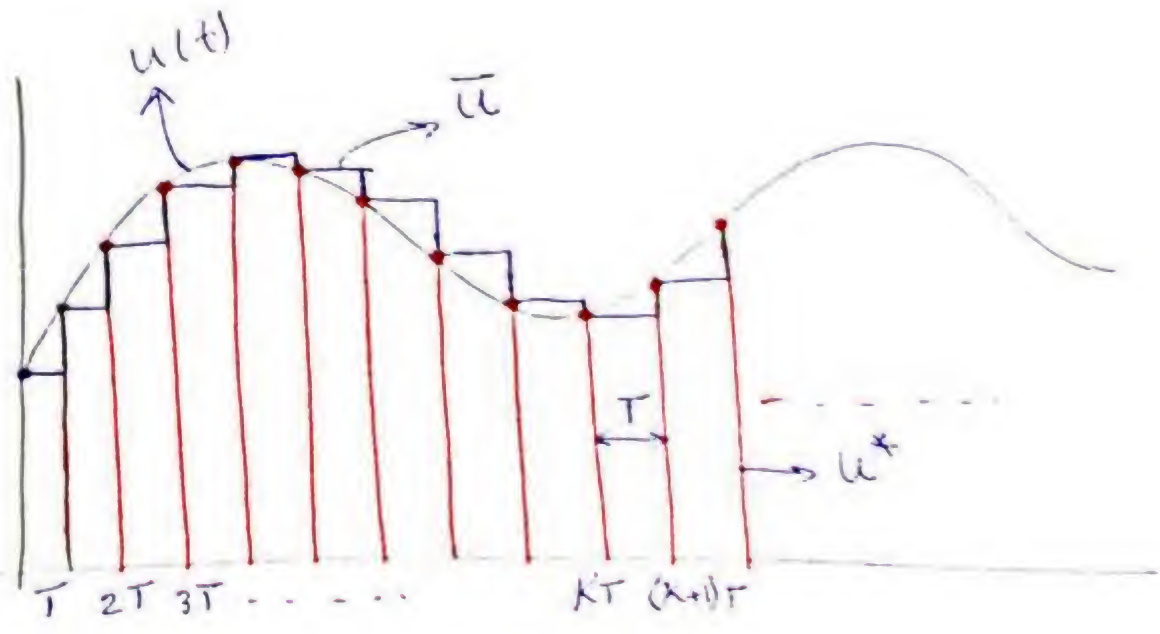
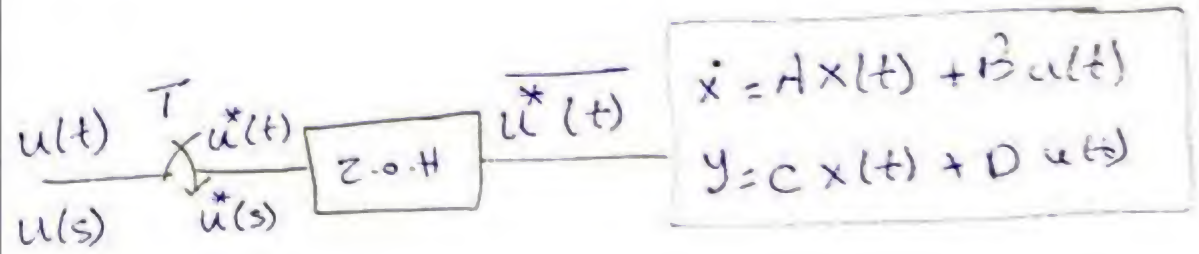
$$D = D_d$$

$$x(t) = \phi(t) x(0) + \int_0^t \phi(t-\tau) B u(\tau) d\tau$$

initial time

initial time t_0

$$x(t) = \phi(t-t_0) x(t_0) + \int_{t_0}^t \phi(t-\tau) B u(\tau) d\tau$$



$t_0 \longrightarrow$ initial value KT

$t \longrightarrow (K+1)T$

$$\phi(t-t_0) = KT + \frac{T}{K} - KT = T$$

$$x((k+1)T) = \phi(T)x(kT) + \int_{\tau=kT}^{(k+1)T} \phi((k+1)T - \tau) B \underbrace{u(\tau)}_{\text{Const.}} d\tau$$

$$x((k+1)T) = \phi(T)x(kT) + u(kT) \int_{\tau=kT}^{(k+1)T} \underbrace{\phi((k+1)T - \tau) B}_{\gamma} d\tau$$

assume $\boxed{(k+1)T - \tau = \gamma} \rightarrow (1)$

$$\frac{d\gamma}{d\tau} = -1 \quad \leadsto \quad \boxed{d\tau = -d\gamma}$$

in (1) $\tau = kT \Rightarrow \gamma = T$

$\tau = (k+1)T \Rightarrow \gamma = 0$

$$x((k+1)T) = \phi(T)x(kT) + u(kT) \int_T^0 \phi(\gamma) B * -d\gamma$$

$$x(k+1)T = \phi(T) x(kT) + u(kT) \int_0^T \phi(\tau) \cdot B \cdot d\tau$$

Compare with

$$x(k+1)T = A_d x(kT) + B_d u(kT)$$

$$A_d = \phi(T)$$

$$B_d = \int_0^T \phi(\tau) \cdot B \cdot d\tau$$

~~≠~~

→ Summary

$$\dot{x}(t) = A x(t) + B u(t)$$

$$y(t) = C x(t) + D u(t)$$

Discretization
(sampler + Z.o.H)

$$x(k+1)T = A_d x(kT) + B_d u(kT)$$

$$y(kT) = C_d x(kT) + D_d u(kT)$$

$$A_d = \phi(T)$$

$$C_d = C$$

$$D_d = D$$

$$B_d = \int_0^T \phi(q) \cdot B \cdot dq$$

where:

$$\phi(T) = \phi(t) \Big|_{t=T}$$

$$\phi(t) = \mathcal{L}^{-1} (sI - A)^{-1}$$

$$\phi(q) = \phi(t) \Big|_{t=q}$$

EX Find the discrete state-space for:

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)$$

$$y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(t)$$

Sampler and Z-o-H are used
with $T = 0.1$ sec.

$$(sI - A) = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} s & -1 \\ 0 & s+1 \end{pmatrix}$$

$$(sI - A)^{-1} = \frac{1}{s(s+1)} \begin{pmatrix} s+1 & 1 \\ 0 & s \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{s} & \frac{1}{s(s+1)} \\ 0 & \frac{1}{s+1} \end{pmatrix}$$

$$(sI - A)^{-1} = \begin{pmatrix} \frac{1}{s} & \frac{1}{s} - \frac{1}{s+1} \\ 0 & \frac{1}{s+1} \end{pmatrix}$$

$$\phi(t) = \mathcal{L}^{-1} (sI - A)^{-1} = \begin{pmatrix} 1 & 1 - e^{-t} \\ 0 & e^{-t} \end{pmatrix}$$

↳ state-transition matrix

$$A_d = \phi(T) = \phi(t) \big|_{t=T}$$

$$= \begin{pmatrix} 1 & 1 - e^{-T} \\ 0 & e^{-T} \end{pmatrix}$$

$$A_d = \begin{pmatrix} 1 & 1 - e^{-0.1} \\ 0 & e^{-0.1} \end{pmatrix} = \begin{pmatrix} 1 & 0.0952 \\ 0 & 0.905 \end{pmatrix}$$

$$b_d = \int_0^T \phi(\tau) B d\tau$$

$$= \int_0^{0.1} \begin{pmatrix} 1 & 1 - e^{-\tau} \\ 0 & e^{-\tau} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} d\tau$$

$$= \int_0^{0.1} \begin{pmatrix} 1 - e^{-\tau} \\ e^{-\tau} \end{pmatrix} d\tau = \begin{pmatrix} \tau + e^{-\tau} \\ -e^{-\tau} \end{pmatrix} \Big|_0^{0.1}$$

$$b_d = \begin{pmatrix} 0.1 + e^{-0.1} \\ -e^{-0.1} \end{pmatrix} - \begin{pmatrix} 0 + 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0.005 \\ 0.095 \end{pmatrix}$$

$$c_d = c = (1 \quad 0)$$

$$x(k+1)T = \begin{pmatrix} 1 & 0.0952 \\ 0 & 0.905 \end{pmatrix} x(kT) + \begin{pmatrix} 0.005 \\ 0.095 \end{pmatrix} u(kT)$$

$$y(kT) = (1 \quad 0) x(kT)$$

T.F



$$\text{Pulse T.F} = C_d (zI - A_d)^{-1} B_d + D_d$$

or

$$T.F = C (sI - A)^{-1} B + D = F(s)$$

$$\text{Pulse T.F} = \sum_{n=0}^{\infty} \left[\frac{1 - e^{-Ts}}{s} \cdot F(s) \right]$$

* state-Feedback Control
↳ (Pole Placement design)

Control design

Classical Control Design

- P.I

- P.D

- P.I.D

- Phase lead, Phase lag...

modern Control design

↳ state Feedback Control

observer design

Control Problem

regulation control
problem

$(r=0)$

→ we concern with
the disturbance or
noise rejection.

→ there is no input.

ex
state feedback
control.

Servo Problem

$(r \neq 0)$

→ we concern with
enhancement of the
system performance
in the presence of
a desired reference
input

→ input is existed.

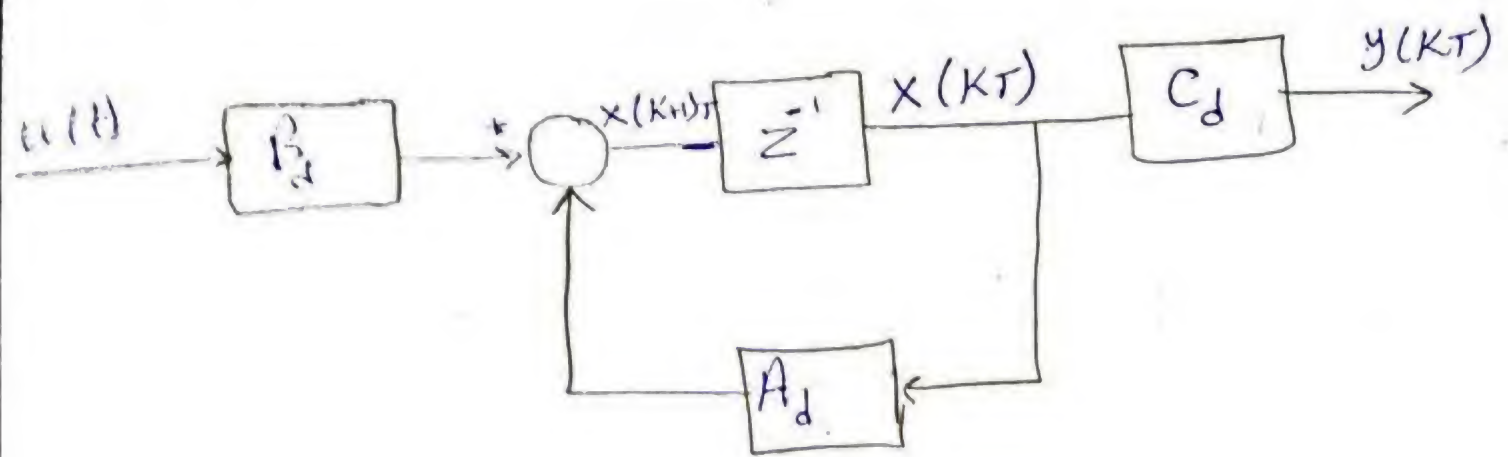
→ For a discrete time system

~~$x(kT)$~~

$$x(k+1)T = A x(kT) + B u(t)$$

$$y(kT) = C x(kT)$$

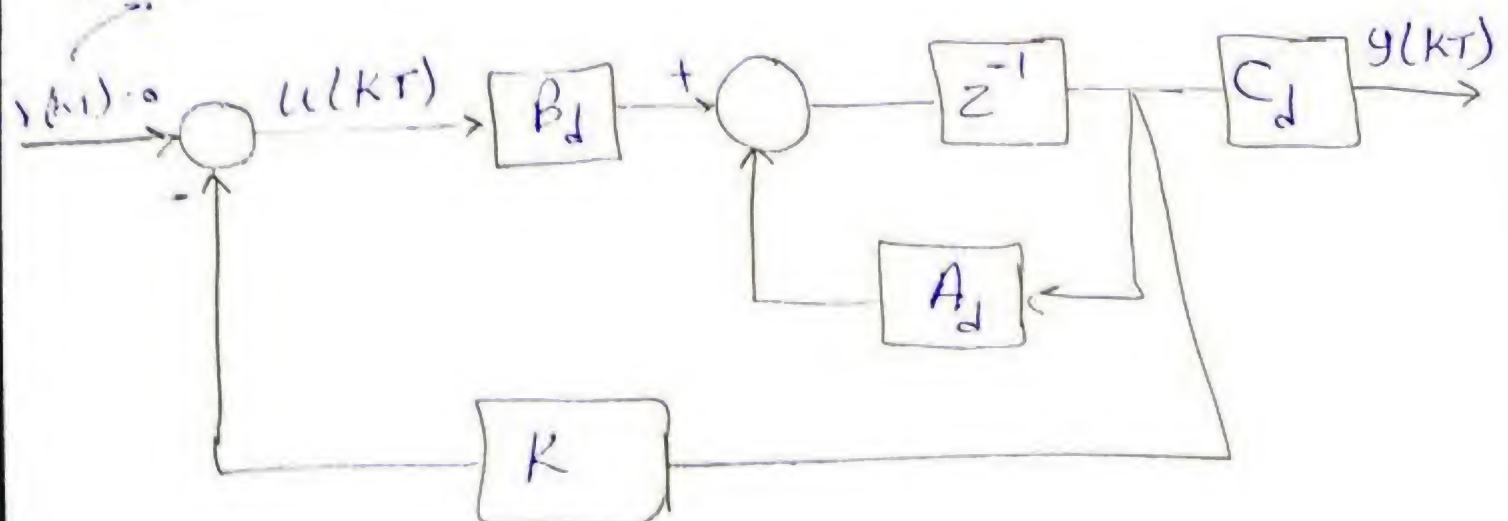
$A, B, C \rightarrow$ in discrete time system.



~~using~~ using state feedback control ::

$$u(k) = -K x(k)$$

regulation Problem



→ The design problem for the state feedback control is to find the gain matrix K

$$\text{where } K = [K_1 \quad K_2 \quad \dots \quad K_n]$$

where n is system order.

Given the desired Poles locations, we can obtain the desired ch. equation :-

- assume the desired Poles located at z_1, z_2, \dots

- the desired ch. equation $\alpha_c(z)$

$$\alpha_c(z) = (z - z_1)(z - z_2) \dots = 0$$

$\hookrightarrow \boxed{11}$

using the relation

$$u(KT) = -K x(KT)$$

$$\begin{aligned}
 x(k+1)T &= A_d x(kT) + B_d \underbrace{u(kT)}_{\rightarrow -Kx(kT)} \\
 &= (A_d - B_d K) x(kT)
 \end{aligned}$$

$$\text{ch. eq: } |zI - A_d + B_d K| = 0 \rightarrow \boxed{2}$$

eq. 1 \simeq eq. 2

\rightarrow Compare coefficient to get K

Another method to get the gain matrix K:-

\rightarrow Ackermann's method

$$K = [K_1 \quad K_2 \quad \dots \quad K_n] = (0 \quad 0 \quad \dots \quad 1) M_c^{-1} \alpha_c(A)$$

where:

$$M_c = (B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B) \rightarrow \text{Controllability matrix}$$

$$\alpha_c(A) = \alpha_c(z) \Big|_{z \rightarrow A} \quad \text{where } \alpha_c(z): \text{ desired ch. equation}$$

For $n=2$

$$K = [K_1 \quad K_2] = (0 \quad 1) M_c^{-1} \propto_c(A_d)$$

$$M_c = (B_d \quad A_d B_d)$$

→ if we give the desired poles in s-domain
or in Cont. time system:-

Given desired Specs:-

$$\angle \quad \omega_n$$

$$s_{1,2} = -\zeta \omega_n \pm j \omega_n \sqrt{1 - \zeta^2}$$

$$s = \sigma + j\omega \Rightarrow Z = r \angle \theta$$

$$Z = r \cos \theta \pm j r \sin \theta$$

$$Z_{1,2} = r \angle \pm \theta$$

$$Y = e^{-Z W_n T} \quad \& \quad \Theta = W_n T \sqrt{1 - Z^2}$$

EX
$$X(K+1) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} X(K) + \begin{pmatrix} 2 \\ 2 \end{pmatrix} u(K)$$

$$Y(K) = (1 \quad 0) X(K)$$

→ using state-feedback control to find the gain matrix K such that the desired poles located at ~~there~~

$$Z_{1,2} = 0.528 \pm j 0.295$$

using Ackermann

$$K = (0 \quad 1) M_c^{-1} \alpha_c(A)$$

$$\alpha_c(z) = \underbrace{\left(\underbrace{0.528}_{\substack{\uparrow \\ x}} + \underbrace{j 0.295}_{\substack{\uparrow \\ y}} \right)}_{z-} \left(\underbrace{0.528}_{\substack{\uparrow \\ x}} - \underbrace{j 0.295}_{\substack{\uparrow \\ y}} \right) = 0$$

$$= (z - (0.528)^2) + (0.295)^2 = 0$$

$$\alpha_c(z) = z^2 - 1.056z + 0.366$$

$$\alpha_c(A) = \alpha_c(z) \Big|_{z=A} = A^2 - \cancel{1.056} 1.056A + 0.366I$$

$$\alpha_c(A) = \begin{pmatrix} 0.31 & 1.889 \\ 0 & 0.31 \end{pmatrix}$$

$$M_c = \begin{pmatrix} B_d & A_d B_d \end{pmatrix} \begin{cases} M_c^{-1} = \\ = \frac{1}{-8} \begin{pmatrix} 2 & -6 \\ 2 & 2 \end{pmatrix} \end{cases}$$

$$K = \begin{pmatrix} 0 & 1 \end{pmatrix} M_c^{-1} \alpha_c(A)$$

~~0.0775~~

$$K = (0.0775 \quad 0.3945)$$

Ex

$$X(K+1)T = \begin{pmatrix} 1 & 0.0952 \\ 0 & 0.905 \end{pmatrix} X(KT) + \begin{pmatrix} 0.00434 \\ 0.0952 \end{pmatrix} u(K)$$

$$y(KT) = \begin{pmatrix} 1 & 0 \end{pmatrix} X(KT)$$

→ Find gain matrix K using the Pole Placement design that makes system has a damping ratio $\zeta = 0.46$ & time constant $\tau = 1 \text{ sec}$ & $\frac{1}{T} = 0.1 \text{ sec}$

Given $\zeta = 0.46$

$$\tau = \frac{1}{\zeta \omega_n} = 1 \text{ sec} \Rightarrow \omega_n = 2.174 \text{ rad/sec}$$

$$s_{1,2} = -\zeta \omega_n \pm j \omega_n \sqrt{1 - \zeta^2}$$

$$r = e^{-\zeta \omega_n T} = 0.905$$

$$\theta = \omega_n T \sqrt{1 - \zeta^2} = 0.193 \text{ rad}$$

$$Z_{1,2} = r \cos \theta \pm j r \sin \theta$$

$$Z_{1,2} = 0.888 \pm j 0.1736$$

$$K = (0 \quad 1) M_c^{-1} \alpha_c(A)$$

$$M_c = (B_d \quad A_d B_d) = \begin{pmatrix} 0.00484 & 0.0137 \\ 0.0952 & 0.0862 \end{pmatrix}$$

$$M_c^{-1} = \begin{pmatrix} -95.035 & 15.34 \\ 105.1 & -5.34 \end{pmatrix}$$

$$\alpha_c(z) = z^2 - 1.776z + 0.819 = 0$$

$$\alpha_c(A) = \begin{pmatrix} 0.043 & 0.0123 \\ 0 & 0.0307 \end{pmatrix}$$

$$K = (4.515 \quad 1.125)$$